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Periodic and localized solutions of the long wave–short wave resonance interaction equation

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Abstract
In this paper, we investigate the (2+1)-dimensional long wave–short wave resonance interaction (LSRI) equation and show that it possess the Painlevé property. We then solve the LSRI equation using Painlevé truncation approach through which we are able to construct solution in terms of three arbitrary functions. Utilizing the arbitrary functions present in the solution, we have generated a wide class of elliptic function periodic wave solutions and exponentially localized solutions, such as dromions, multidromions, instantons, multi-instantons and bounded solitary wave solutions.

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1. Introduction

The identification of dromions which are exponentially decaying solutions in the Davey–Stewartson I and other equations [1–6] has triggered a renewed interest in the study of integrable models in (2+1) dimensions. Dromions arise essentially by virtue of coupling the field variable to a mean field/potential, thereby preventing wave collapse in (2+1) dimensions and they can, in general, undergo inelastic collision unlike one-dimensional solitons. The identification of a large number of arbitrary functions in the solutions of (2+1)-dimensional integrable models has only added to the richness in the structure of them and hence construction of localized excitations in (2+1) dimensions continues to be a challenging and rewarding contemporary problem.

In this paper, we consider the existence of localized structures in the long wave–short wave resonance interaction equation of the form,

$$\mathrm{i}(S_t + S_y) - S_{xx} + LS = 0, \tag{1a}$$

$$L_t = (2SS^*)_x, \tag{1b}$$

where the fields S and L denote short surface wave packets and long interfacial waves respectively, while $*$ stands for the complex conjugation. The above equation has recently been studied [7, 8] and its positon and one dromion solutions have been generated through Hirota method. However, no further general solutions could be constructed through this procedure for equations (1). In this contribution, we develop a very simple and straightforward procedure to generate a rather extended class of generic solutions of physical interest. For this purpose, first we carry out the singularity structure analysis to the LSRI equation and confirm its Painlevé nature. We utilize the local Laurent expansion of the general solution and truncate it at the constant level term (Painlevé truncation approach) and obtain solutions in terms of arbitrary functions. Through this procedure, we generate various periodic and exponentially localized solutions to equation (1). The novelty here is that the solution is generated through a very simple procedure but the solution obtained is rich in structure because of the arbitrary functions [9–14] present in the solution.

The plan of the paper is as follows. In section 2, we present the singularity structure analysis of the LSRI equation. Using these results, in section 3, we have shown the construction of solutions for the LSRI equation through the Painlevé truncation approach. Section 4 contains a wide class of localized solutions of the LSRI equation, both periodic and exponentially localized ones, through a judicial choice of the arbitrary functions. In section 5, we summarize our results. The appendix contains the one dromion solution of LSRI equation obtained through Hirota bilinearization approach for comparison.

2. Singularity structure analysis

To explore the singularity structure of equation (1), we rewrite $S = q$ and $S^* = r$ to obtain the following set of coupled equations:

$$\mathrm{i}(q_t + q_y) - q_{xx} + Lq = 0, \tag{2a}$$

$$-\mathrm{i}(r_t + r_y) - r_{xx} + Lr = 0, \tag{2b}$$

$$L_t = (2qr)_x. \tag{2c}$$

We now effect a local Laurent expansion in the neighbourhood of a noncharacteristic singular manifold $\phi(x, y, t) = 0$, $\phi_x \neq 0$, $\phi_y \neq 0$. Assuming the leading orders of the solutions of equation (2) to have the form

$$q = q_0\phi^\alpha, \qquad r = r_0\phi^\beta, \qquad L = L_0\phi^\gamma, \tag{3}$$

where q_0, r_0 and L_0 are analytic functions of (x, y, t) and α, β, γ are integers to be determined, we substitute (3) into (2) and balance the most dominant terms to obtain

$$\alpha = \beta = -1, \qquad \gamma = -2, \tag{4}$$

with the condition

$$q_0 r_0 = \phi_x\phi_t, \qquad L_0 = 2\phi_x^2. \tag{5}$$

Now, considering the generalized Laurent expansion of the solutions in the neighbourhood of the singular manifold,

$$q = q_0\phi^\alpha + \cdots + q_j\phi^{r+\alpha} + \cdots, \tag{6a}$$

$$r = r_0\phi^{\beta} + \cdots + r_j\phi^{r+\beta} + \cdots, \tag{6b}$$

$$L = L_0\phi^{\gamma} + \cdots + L_j\phi^{r+\gamma} + \cdots, \tag{6c}$$

the resonances which are the powers at which arbitrary functions enter into (6) can be determined by substituting (6) into (2). Vanishing of the coefficients of $(\phi^{j-3}, \phi^{j-3}, \phi^{j-3})$ leads to the condition

$$\begin{pmatrix} -j(j-3)\phi_x^2 & 0 & q_0 \\ 0 & -j(j-3)\phi_x^2 & r_0 \\ 2(j-2)r_0\phi_x & 2(j-2)q_0\phi_x & -(j-2)\phi_x \end{pmatrix} \begin{pmatrix} q_j \\ r_j \\ L_j \end{pmatrix} = 0. \tag{7}$$

From equation (7), one gets the resonance values as

$$j = -1, 0, 2, 3, 4. \tag{8}$$

The resonance at $j = -1$ naturally represents the arbitrariness of the manifold $\phi(x, y, t) = 0$. In order to prove the existence of arbitrary functions at the other resonance values, we now substitute the full Laurent series,

$$q = q_0\phi^{\alpha} + \sum_j q_j\phi^{j+\alpha}, \tag{9a}$$

$$b = r_0\phi^{\beta} + \sum_j r_j\phi^{j+\beta}, \tag{9b}$$

$$L = L_0\phi^{\gamma} + \sum_j L_j\phi^{j+\gamma}, \tag{9c}$$

into equation (2). Now collecting the coefficients of $(\phi^{-3}, \phi^{-3}, \phi^{-3})$ and solving the resultant equation, we obtain equation (5), implying the existence of a resonance at $j = 0$.

Similarly, collecting the coefficients of $(\phi^{-2}, \phi^{-2}, \phi^{-2})$ and solving the resultant equations by using the Kruskal ansatz, $\phi(x, y, t) = x + \psi(y, t)$, we get

$$q_1 = \tfrac{1}{2}[\mathrm{i}q_0(\psi_t + \psi_y) - 2q_{0x}], \tag{10a}$$

$$r_1 = \tfrac{1}{2}[-\mathrm{i}r_0(\psi_t + \psi_y) - 2r_{0x}], \tag{10b}$$

$$L_1 = 0. \tag{10c}$$

Collecting the coefficients of $(\phi^{-1}, \phi^{-1}, \phi^{-1})$, we have

$$\mathrm{i}(q_{0t} + q_{0y}) - q_{0xx} + L_0q_2 + L_1q_1 + L_2q_0 = 0, \tag{11a}$$

$$-\mathrm{i}(r_{0t} + r_{0y}) - r_{0xx} + L_0r_2 + L_1r_1 + L_2r_0 = 0, \tag{11b}$$

$$L_{1t} = 2[q_{0x}r_1 + r_{0x}q_1 + q_{1x}r_0 + q_1r_{0x}] = 0. \tag{11c}$$

From (11a) and (11b), we can eliminate L_2 to obtain a single equation for the two unknowns q_2 and r_2,

$$L_0(r_0q_2 - q_0r_2) - (r_0q_{0xx} - q_0r_{0xx}) + \mathrm{i}(r_0(q_{0t} + q_{0y}) + q_0(r_{0t} + r_{0y})) = 0. \tag{11d}$$

which ensures that either q_2 or r_2 is arbitrary. Obviously, L_2 itself can be obtained either from (11a) or (11b). Similarly, collecting the coefficients of (ϕ^0, ϕ^0, ϕ^0), we have

$$\mathrm{i}(q_{1t} + q_2\psi_t) + \mathrm{i}(q_{1y} + q_2\psi_y) - (q_{1xx} + 2q_{2x}) + L_2q_1 + L_3q_0 = 0, \tag{12a}$$

$$-\mathrm{i}(r_{1t} + r_2\psi_t) - \mathrm{i}(r_{1y} + r_2\psi_y) - (r_{1xx} + 2r_{2x}) + L_2r_1 + L_3r_0 = 0, \tag{12b}$$

$$L_{2t} + L_3\psi_t = 2[q_{0x}r_2 + (q_{1x} + q_2)r_1 + (q_{2x} + 2q_3)r_0 + r_{0x}q_2 + (r_{1x} + r_2)q_1 + (r_{2x} + 2r_3)q_0]. \quad (12c)$$

We rewrite equations (12*a*) and (12*b*) as

$$L_3 = \frac{1}{q_0}(-\mathrm{i}(q_{1t} + q_2\psi_t) - \mathrm{i}(q_{1y} + q_2\psi_y) + (q_{1xx} + 2q_{2x}) - L_2q_1), \quad (12d)$$

$$L_3 = \frac{1}{r_0}(+\mathrm{i}(r_{1t} + r_2\psi_t) + \mathrm{i}(r_{1y} + r_2\psi_y) + (r_{1xx} + 2r_{2x}) - L_2r_1). \quad (12e)$$

Making use of the earlier relations (5), (10*b*) and (11*b*), we find that the right-hand sides of equations (12*d*) and (12*e*) are equal. Then, we are left with two equations for three unknowns. So, one of the three coefficients q_3, r_3 or L_3 is arbitrary. Collecting now the coefficients of (ϕ, ϕ, ϕ), we have

$$\mathrm{i}(q_{2t} + 2q_3\psi_t) + \mathrm{i}(q_{2y} + 2q_3\psi_y) - (q_{2xx} + 4q_{3x} + 6q_4) + L_0q_4 + L_2q_2 + L_3q_1 + L_4q_0 = 0, \quad (13a)$$

$$-\mathrm{i}(r_{2t} + 2r_3\psi_t) - \mathrm{i}(r_{2y} + 2r_3\psi_y) - (r_{2xx} + 4r_{3x} + 6r_4) + L_0r_4 + L_2r_2 + L_3r_1 + L_4r_0 = 0, \quad (13b)$$

$$L_{3t} + 2L_4\psi_t = 2[q_{0x}r_3 - q_0r_4 + (q_{1x} + q_2)r_2 + (q_{2x} + 2q_3)r_1 + (q_{3x} + 3q_4)r_0 + r_{0x}q_3 - r_0q_4 + (r_{1x} + r_2)q_2 + (r_{2x} + 2r_3)q_1 + (r_{3x} + 3r_4)q_0]. \quad (13c)$$

Here also, the above set of three equations reduces to two equations. So, one of the three functions q_4, r_4 or L_4 is arbitrary. One can proceed further to determine all other coefficients of the Laurent expansions (9) without the introduction of any movable critical singular manifold. Thus, the LSRI equation indeed satisfies the Painlevé property.

3. Painlevé truncation approach

To generate the solutions of LSRI equation, we suitably harness the results of the Painlevé analysis. Truncating the Laurent series of the solutions of the LSRI equation at the constant level term, we have the Bäcklund transformation

$$q = \frac{q_0}{\phi} + q_1, \quad (14a)$$

$$r = \frac{r_0}{\phi} + r_1, \quad (14b)$$

$$L = \frac{L_0}{\phi^2} + \frac{L_1}{\phi} + L_2. \quad (14c)$$

Assuming a seed solution given by

$$q_1 = r_1 = 0, \qquad L_2 = L_2(x, y), \quad (15)$$

we now substitute (14) with the above seed solution (15) into equations (2) to obtain the following system of equations by equating the coefficients of $(\phi^{-3}, \phi^{-3}, \phi^{-3})$:

$$-2q_0\phi_x^2 + L_0q_0 = 0, \quad (16a)$$

$$-2r_0\phi_x^2 + L_0r_0 = 0, \quad (16b)$$

$$L_0\phi_t = 2q_0r_0\phi_x. \tag{16c}$$

Solving the above system of equations, we obtain the leading order coefficients already given by equation (5), namely $q_0r_0 = \phi_x\phi_t$ and $L_0 = 2\phi_x^2$. Now collecting the coefficients $(\phi^{-2}, \phi^{-2}, \phi^{-2})$ we have the following system of equations:

$$-\mathrm{i}q_0\phi_t - \mathrm{i}q_0\phi_y + 2q_{0x}\phi_x + q_0\phi_{xx} + L_1q_0 = 0, \tag{17a}$$

$$\mathrm{i}r_0\phi_t + \mathrm{i}r_0\phi_y + 2r_{0x}\phi_x + r_0\phi_{xx} + L_1r_0 = 0, \tag{17b}$$

$$L_{0t} - L_1\phi_t = 2(q_0r_0)_x. \tag{17c}$$

From equation (17c), we have

$$L_1 = -2\left[\phi_{xx} + \frac{\phi_x\phi_{tx}}{\phi_t}\right] \tag{18}$$

Using (18) in equation (17a) or (17b), one can easily obtain the relation

$$\frac{q_{0x}}{q_0} = \frac{1}{2}\left[\frac{\mathrm{i}(\phi_t + \phi_y) + \phi_{xx} - \frac{2\phi_x\phi_{tx}}{\phi_t}}{\phi_x}\right] \tag{19}$$

On integration, we obtain

$$q_0 = F(y,t)\exp\left[\frac{1}{2}\int \frac{\mathrm{i}(\phi_t + \phi_y) + \phi_{xx} - \frac{2\phi_x\phi_{tx}}{\phi_t}}{\phi_x}\,\mathrm{d}x\right], \tag{20}$$

where $F(y,t)$ is an arbitrary function of y and t. Obviously the above solution is consistent with (17).

Again collecting the coefficients of $(\phi^{-1}, \phi^{-1}, \phi^{-1})$, we have the following set of equations:

$$\mathrm{i}q_{0t} + \mathrm{i}q_{0y} - q_{0xx} + L_2q_0 = 0, \tag{21a}$$

$$-\mathrm{i}r_{0t} - \mathrm{i}r_{0y} - r_{0xx} + L_2r_0 = 0, \tag{21b}$$

$$L_{1t} = 0. \tag{21c}$$

Using (18), we rewrite equation (21c) to obtain the trilinear form

$$\phi_t^2\phi_{xxt} - \phi_x\phi_{tx}\phi_{tt} + \phi_{xt}^2\phi_t + \phi_x\phi_{ttx}\phi_t = 0. \tag{22}$$

The structure of the trilinear equation (22) suggests that a specific solution can be given in the form

$$\phi = \phi_1(x,y) + \phi_2(y,t), \tag{23}$$

where $\phi_1(x,y)$ and $\phi_2(y,t)$ are arbitrary functions in the indicated variables. Using (23) in equations (18) and (20), one can obtain the functions q_0 and L_1 as

$$q_0 = F(y,t)\exp\left[\frac{1}{2}\int \frac{\mathrm{i}(\phi_{2t} + \phi_{1y} + \phi_{2y}) + \phi_{1xx}}{\phi_{1x}}\,\mathrm{d}x\right], \tag{24a}$$

$$L_1 = -2\phi_{1xx}. \tag{24b}$$

From (24b), we find that equation (21c) is an identity. Using (24a), equations (21a) and (21b) can be reduced to the form

$$\phi_{2tt} + \phi_{2ty} = 0. \tag{25}$$

Equation (25) can be solved readily to express the submanifold $\phi_2(y,t)$ in the form

$$\phi_2 = F_2(y) + F_3(t-y), \tag{26}$$

where $F_2(y)$ and $F_3(t-y)$ are arbitrary functions in y and $(t-y)$, respectively.

Finally, collecting the coefficients of (ϕ^0, ϕ^0, ϕ^0), we have only one equation

$$L_{2t} = 0. \tag{27}$$

Using (21*a*) for L_2, (27) reduces to the form

$$(F_{tt} + F_{ty})F + (F_t + F_y)F_t = 0. \tag{28}$$

Equation (28) can be solved to obtain the form for $F(y,t)$ as

$$F(y,t) = F_1(t-y). \tag{29}$$

Thus the LSRI equation (1) has been solved by the truncated Painlevé approach and the fields q and r can be given in terms of the arbitrary functions as

$$q = \frac{q_0}{\phi_1(x,y) + F_2(y) + F_3(t-y)}, \tag{30a}$$

$$r = \frac{\phi_{1x}\phi_{2t}}{q_0(\phi_1(x,y) + F_2(y) + F_3(t-y))} \tag{30b}$$

and

$$L = \frac{2\phi_{1x}^2}{(\phi_1(x,y) + F_2(y) + F_3(t-y))^2} - \frac{2\phi_{1xx}}{(\phi_1(x,y) + F_2(y) + F_3(t-y))} + L_2, \tag{31}$$

where

$$L_2 = \int \frac{1}{2}\left(\frac{\mathrm{i}(\phi_{1yy} + F_{2yy}) + \phi_{1xxy}}{\phi_{1x}} - \frac{\mathrm{i}(\phi_{1y} + F_{2y}) + \phi_{1xx}}{\phi_{1x}^2}\phi_{1xy}\right)\mathrm{d}x + \frac{1}{2}\frac{\mathrm{i}\phi_{1xy} + \phi_{1xxx}}{\phi_{1x}} - \frac{1}{4}\frac{(\phi_{1y} + F_{2y})^2 + \phi_{1xx}^2}{\phi_{1x}^2}. \tag{32}$$

Here the function $\phi_2(y,t)$ is given by equation (26) and q_0 by (24*a*), while the functions $\phi_1(x,y)$, $F_2(y)$, $F_3(t-y)$ are themselves arbitrary in the indicated variables.

4. Novel exact solutions of LSRI equation

Now we make use of the above-truncated Laurent expansion solution to obtain exact solutions of the LSRI equation (1) for the variables $S(x,y,t)$ and $L(x,y,t)$.

Taking into account our notation in equation (2), that is $q = S(x,y,t)$ and $r = S^*(x,y,t)$, we have $q = r^*$ as far as equation (1) is concerned. Using this condition in equations (30*a*) and (30*b*), we obtain the condition

$$[F_1(t-y)]^2 = F_{3t}. \tag{33}$$

Thus, from the results of the previous section, we find that the solution of the original variable $S(x,y,t)$ takes the form

$$S(x,y,t) = \frac{\sqrt{F_{3t}\phi_{1x}}\exp\left(\int \frac{1}{2}\frac{\mathrm{i}(\phi_{1y}+F_{2y})}{\phi_{1x}}\,\mathrm{d}x\right)}{(\phi_1(x,y) + F_2(y) + F_3(t-y))}, \tag{34}$$

while its squared magnitude takes the form

$$|S|^2 = \frac{\phi_{1x}F_{3t}}{(\phi_1(x,y) + F_2(y) + F_3(t-y))^2}. \tag{35}$$

The form of $L(x, y, t)$ remains the same as given in equation (31). With the above general form of the solutions, we now identify interesting classes of exact solutions to equation (1), including periodic and localized solutions by giving specific forms for the three arbitrary functions $\phi_1(x, y)$, $F_2(y)$ and $F_3(t - y)$.

4.1. Periodic solutions and localized dromion solutions

Let us now choose the arbitrary functions ϕ_1 and F_3 to be Jacobian elliptic functions, namely sn or cn functions. The motivation behind this choice of arbitrary function stems from the fact that the limiting forms of these functions happen to be localized functions. Hence, a choice of cn and sn functions can yield periodic solutions which are more general than exponentially localized solutions (dromions). We choose, for example,

$$\phi_1 = \mathrm{sn}(ax + by + c_1; m_1), \qquad F_2 = 4, \qquad F_3 = \mathrm{sn}(t - y + c_2; m_2), \tag{36}$$

so that

$$S(x, y, t) = \frac{\sqrt{a}\,\mathrm{cn}(u_1; m_1)\,\mathrm{dn}(u_1; m_1)\,\mathrm{cn}(u_2; m_2)\,\mathrm{dn}(u_2; m_2)}{(4 + \mathrm{sn}(u_2; m_2) + \mathrm{sn}(u_1; m_1))}\,\mathrm{e}^{\frac{ib}{2a}x}, \tag{37}$$

where $u_1 = ax + by + c_1$ and $u_2 = t - y + c_2$. In equations (36) and (37), the quantities m_1 and m_2 are the modulus parameters of the respective Jacobian elliptic functions while a, b, c_1 and c_2 are arbitrary constants. The corresponding expression for $|S(x, y, t)|^2$ takes the form

$$|S|^2 = \frac{|a\,\mathrm{cn}(u_1; m_1)\,\mathrm{dn}(u_1; m_1)\,\mathrm{cn}(u_2; m_2)\,\mathrm{dn}(u_2; m_2)|}{(4 + \mathrm{sn}(u_2; m_2) + \mathrm{sn}(u_1; m_1))^2}. \tag{38}$$

The profile of the above solution for the parametric choice $a = b = 1$, $c_1 = c_2 = 0$, $m_1 = 0.2$, $m_2 = 0.3$, $t = 0$ is shown in figure 1(a). Note that the periodic wave moves with unit phase velocity.

4.1.1. (1, 1) dromion solution. As a limiting case of the periodic solution given by equation (38), when $m_1, m_2 \to 1$, the above solution degenerates into an exponentially localized solution (dromion). Noting that $\mathrm{cn}(u; 1) = \mathrm{dn}(u; 1) = \mathrm{sech}\,u$ and $\mathrm{sn}(u; 1) = \tanh u$, the limiting forms corresponding to (1, 1) dromion take the expressions

$$S = \frac{\sqrt{a}\,\mathrm{sech}(t - y + c_2)\,\mathrm{sech}(ax + by + c_1)}{4 + \tanh(ax + by + c_1) + \tanh(t - y + c_2)}\,\mathrm{e}^{\frac{ib}{2a}x} \tag{39}$$

and

$$|S|^2 = \frac{a\,\mathrm{sech}^2(t - y + c_2)\,\mathrm{sech}^2(ax + by + c_1)}{(4 + \tanh(ax + by + c_1) + \tanh(t - y + c_2))^2}. \tag{40}$$

The variable L then takes the form (using expression (31))

$$\begin{aligned} L = {} & \frac{2a^2\,\mathrm{sech}^4(ax + by + c_1)}{(4 + \tanh(ax + by + c_1) + \tanh(t - y + c_2))^2} \\ & - \frac{2\,\mathrm{sech}^2(t - y + c_2)}{(4 + \tanh(ax + by + c_1) + \tanh(t - y + c_2))} \\ & - \tanh(ax + by + c_1) - i\tanh(ax + by + c_1) \\ & + \tanh^2(ax + by + c_1) - \mathrm{sech}^2(ax + by + c_1) - \frac{1}{4} \end{aligned} \tag{41}$$

A schematic form of the (1, 1) dromion for the parametric choice $a = b = 1$, $c_1 = c_2 = 0$ is shown in figure 1(b). Again note that the dromion travels with a unit velocity in a diagonal

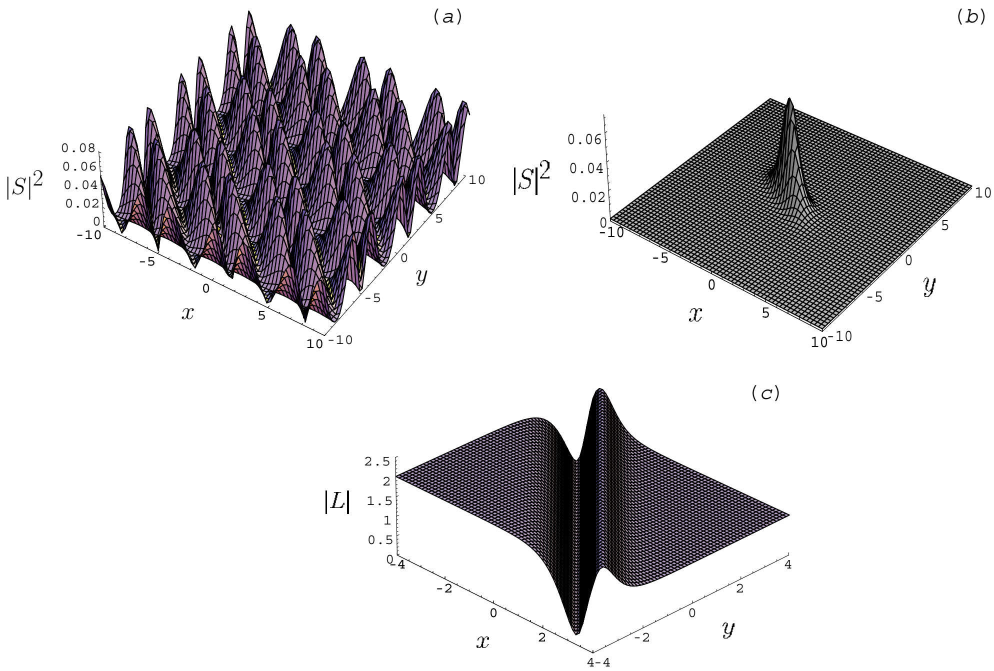

Figure 1. (*a*) Elliptic function solution (38), (*b*) localized dromion solution (40) for the variable $|S(x, y, t)|^2$ and (*c*) the corresponding magnitude of the variable $L(x, y, t)$ given by (41).

direction in the x–y plane. One can check that the (1, 1) dromion obtained by Lai and Chow in [8] using the Hirota bilinear method is a special case of the above solution (40) by fixing the parameters a, b, c_1 and c_2 suitably. However, the later method is unable to give more general solutions (see also the appendix).

4.2. More general periodic solutions and higher order dromion solutions

4.2.1. Periodic solution and (2, 1) dromion. Next we obtain a more general periodic solution by choosing further general forms for the arbitrary functions. As an example, we choose

$$\begin{aligned} \phi_1 &= d_1\,\mathrm{sn}(c_1 + a_1 x + b_1 y; m_1) + d_2\,\mathrm{sn}(c_2 + a_2 x + b_2 y; m_2), \\ F_2 &= 4, \qquad F_3 = d_3\,\mathrm{sn}(c_3 + t - y; m_3), \end{aligned} \tag{42}$$

where a_i, b_i, c_i and d_i are arbitrary constants and m_i's are modulus parameters ($i = 1, 2, 3$). Then

$$|S|^2 = \frac{q_1}{q_2}, \tag{43}$$

where $q_1 = |(d_1 a_1\,\mathrm{cn}\,(u_1; m_1)\,\mathrm{dn}(u_1; m_1) + d_2 a_2\,\mathrm{cn}(u_2; m_2)\,\mathrm{dn}(u_2; m_2))d_3\,\mathrm{cn}(u_3; m_3)\,\mathrm{dn}(u_3; m_3)|$, $q_2 = (4 + d_1\,\mathrm{sn}(u_1; m_1) + d_2\,\mathrm{sn}(u_2; m_2) + d_3\,\mathrm{sn}(u_3; m_3))^2$, $u_1 = c_1 + a_1 x + b_1 y$, $u_2 = c_2 + a_2 x + b_2 y$ and $u_3 = c_3 + t - y$ with corresponding expressions for $S(x, y, t)$. The profile of the above solution for the parametric choice $a_1 = 1, b_1 = 1, a_2 = 1, b_2 = -1, d_1 = 5$, $d_2 = 4$, $d_3 = 0.5$, $c_1 = 0$, $c_2 = c_3 = 5$, $m_1 = 0.2$, $m_2 = 0.3$, $m_3 = 0.4$, $t = 0$ is shown in

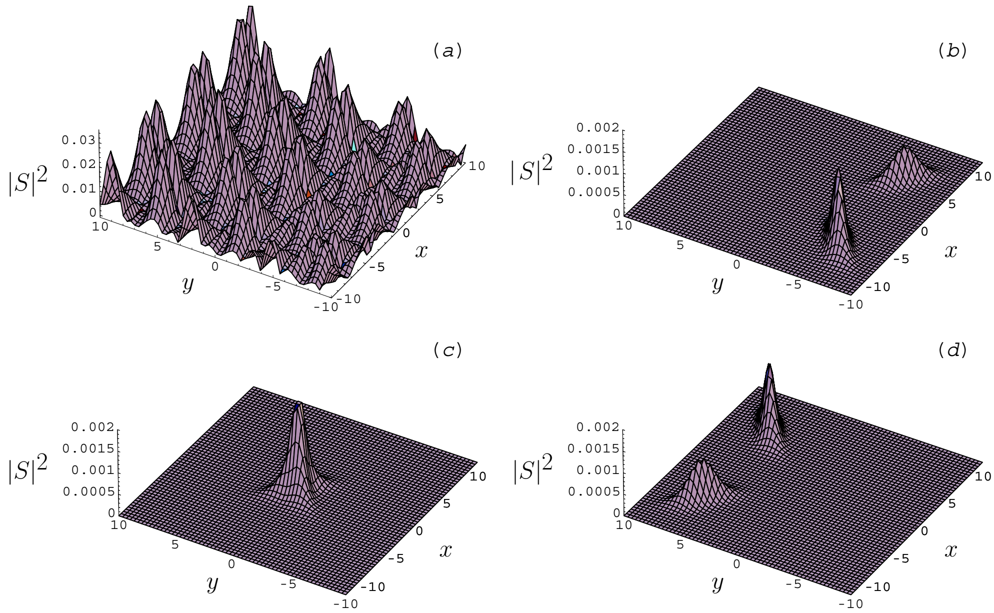

Figure 2. (*a*) Elliptic function solution (43), (*b*)–(*d*) (2, 1) dromion solution (44) and its interaction at time intervals (*b*) $t = -10$, (*c*) $t = 0$ and (*d*) $t = 10$.

figure 2(*a*). As $m_1, m_2, m_3 \to 1$, the above solution, namely equation (43), degenerates into a (2, 1) dromion solution given by

$$|S|^2 = \frac{(d_1 a_1 \operatorname{sech}^2 u_1 + d_2 a_2 \operatorname{sech}^2 u_2) d_3 \operatorname{sech}^2 u_3}{(4 + d_1 \tanh u_1 + d_2 \tanh u_2 + d_3 \tanh u_3)^2} \tag{44}$$

where $u_1 = c_1 + a_1 x + b_1 y$, $u_2 = c_2 + a_2 x + b_2 y$ and $u_3 = c_3 + t - y$. The dromion interaction for the parametric choice $a_1 = b_1 = a_2 = 1$, $b_2 = -1$, $d_1 = 0.5$, $d_2 = d_3 = 1$, $c_1 = c_2 = c_3 = 0$ is shown in figures 2(*b*)–(*d*) for different time intervals. Here both the dromions travel with equal velocity but along opposite diagonals in the *x*–*y* plane. The interaction is elastic for this choice. The variable L can be evaluated again using equation (31), which we desist from presenting here.

4.2.2. Periodic solution and (2, 2) dromion. Another example for more general periodic solution is given by choosing

$$\begin{aligned} \phi_1 &= d_1 \operatorname{sn}(c_1 + a_1 x + b_1 y; m_1) + d_2 \operatorname{sn}(c_2 + a_2 x + b_2 y; m_2), \\ F_2 &= 4, \qquad F_3 = d_3 \operatorname{sn}(c_3 + t - y; m_3) + d_4 \operatorname{sn}(c_4 + t - y; m_4). \end{aligned} \tag{45}$$

In equation (45), we choose $m_1, m_2, m_3 \to 1$, to obtain (2, 2) dromion solution given by

$$|S|^2 = \frac{(d_1 a_1 \operatorname{sech}^2 u_1 + d_2 a_2 \operatorname{sech}^2 u_2)(d_3 \operatorname{sech}^2 u_3 + d_4 \operatorname{sech}^2 u_4)}{(4 + d_1 \tanh u_1 + d_2 \tanh u_2 + d_3 \tanh u_3 + d_4 \tanh u_4)^2}, \tag{46}$$

where $u_1 = c_1 + a_1 x + b_1 y$, $u_2 = c_2 + a_2 x + b_2 y$, $u_3 = c_3 + t - y$ and $u_4 = c_4 + t - y$. The solution of (2, 2) dromion for the parametric choice $a_1 = b_1 = a_2 = b_2 = 1$, $d_1 = 0.5$, $d_2 = d_3 = d_4 = 1$, $c_1 = c_2 = c_3 = c_4 = 0$ is plotted in figure 3 for various time intervals. We find that there are two sets of dromions, each set containing two dromions one followed by the other. The two sets of dromions are travelling with the same velocity in opposite diagonals of the *x*–*y* plane. The dromions interact and move forward as time progresses.

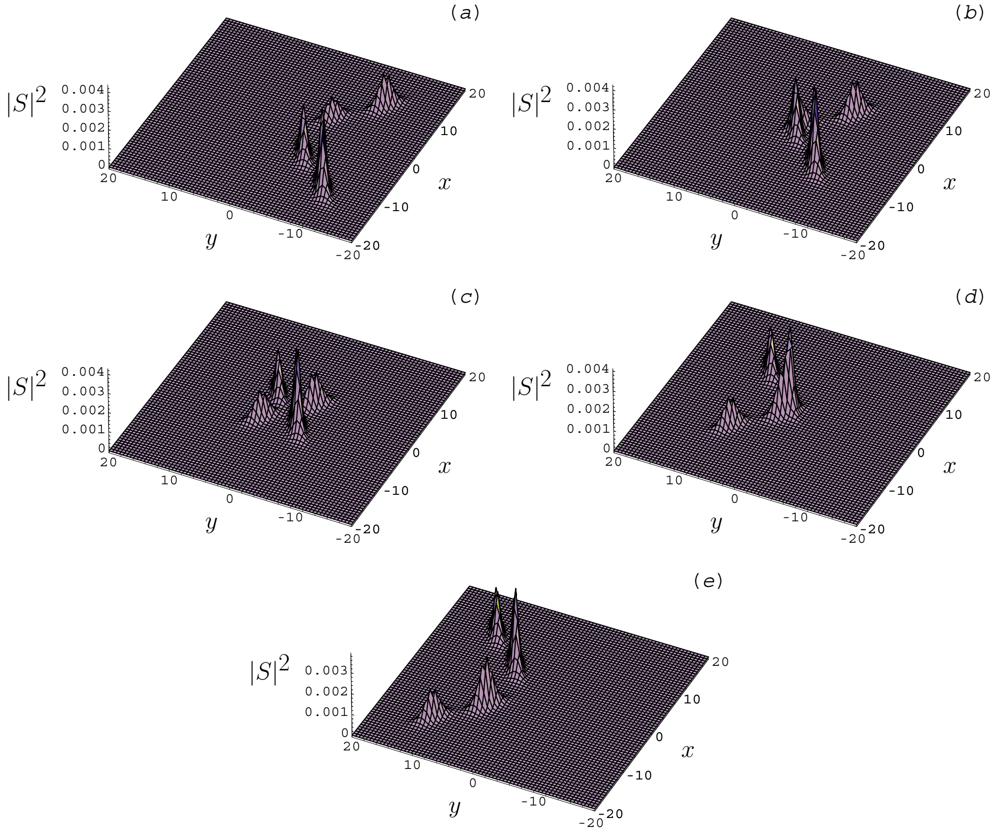

Figure 3. (*a*)–(*e*) (2, 2) dromion solution (46) interaction at time intervals (*a*) $t = -8$, (*b*) $t = -4$, (*c*) $t = 0$, (*d*) $t = 4$ and (*e*) $t = 8$.

4.2.3. (M, N) dromion. To generalize the above solutions, we choose

$$\phi_1 = \sum_{j=1}^{M} d_j \,\mathrm{sn}(c_j + a_j x + b_j y; m_j), \tag{47a}$$

$$F_2 = 4, \qquad F_3 = \sum_{k=1}^{N} d_k \,\mathrm{sn}(c_k + t - y; m_k), \tag{47b}$$

where $a_j, b_j, c_j, d_j, c_k, d_k$ are arbitrary constants and all m_j's and m_k's take values between 0 and 1 for periodic solutions and equal to 1 for dromion solutions. We proceed as above to construct periodic and dromion solutions, respectively, as

$$|S|^2 = \frac{\left|\sum_{j=1}^{M} d_j a_j \,\mathrm{cn}(u_1; m_j)\,\mathrm{dn}(u_1; m_j) \sum_{k=1}^{N} d_k \,\mathrm{cn}(u_2; m_k)\,\mathrm{dn}(u_2; m_k)\right|}{4 + \sum_{j=1}^{M} d_j \,\mathrm{sn}(u_1; m_j) + \sum_{k=1}^{N} d_k \,\mathrm{sn}(u_2; m_k)} \tag{48}$$

and

$$|S|^2 = \frac{\sum_{j=1}^{M} d_j a_j \,\mathrm{sech}^2(c_j + a_j x + b_j y) \sum_{k=1}^{N} d_k \,\mathrm{sech}^2(c_k + t - y)}{4 + \sum_{j=1}^{M} d_j \tanh(c_j + a_j x + b_j y) + \sum_{k=1}^{N} d_k \tanh(c_k + t - y)}, \tag{49}$$

where $u_1 = c_j + a_j x + b_j y$ and $u_2 = c_k + t - y$.

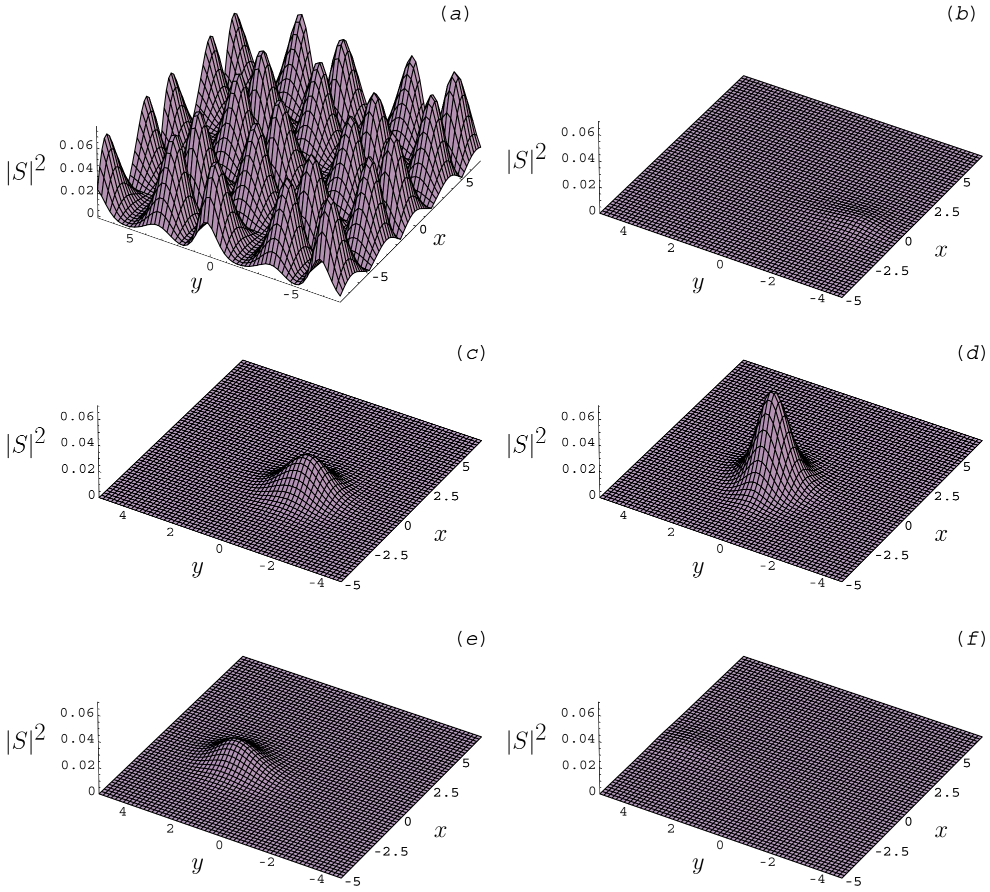

Figure 4. (*a*) Elliptic function solution (51) and an instanton solution (52) at time intervals (*b*) $t = -3$, (*c*) $t = -1$, (*d*) $t = 1$, (*e*) $t = 3$ and (*f*) $t = 4$.

4.3. Instanton-type solutions

Another type of elliptic function solution can be chosen as

$$\phi_1 = \mathrm{sn}(ax + c_1; m_1)\,\mathrm{cn}(by + c_2; m_2), \qquad F_2 = 4, \qquad F_3 = \mathrm{sn}(t - y + c_3; m_3). \tag{50}$$

Then

$$|S|^2 = \frac{|a\,\mathrm{cn}(u_1; m_1)\,\mathrm{dn}(u_1; m_1)\,\mathrm{cn}(u_2; m_2)\,\mathrm{cn}(u_3; m_3)\,\mathrm{dn}(u_3; m_3)|}{(4 + \mathrm{sn}(u_1; m_1)\,\mathrm{cn}(u_2; m_2) + \mathrm{sn}(u_3; m_3))^2}. \tag{51}$$

where $u_1 = ax + c_1$, $u_2 = by + c_2$ and $u_3 = t - y + c_3$. The profile of the above periodic solution for the parametric choices $a = 1$, $b = -1$, $c_1 = c_2 = c_3 = 0$, $m_1 = 0.2$, $m_2 = 0.3$, $m_3 = 0.4$ is shown in figure 4(*a*).

As $m_1, m_2, m_3 \to 1$, equation (51) degenerates into an instanton-type solution,

$$|S|^2 = \frac{a\,\mathrm{sech}^2(t - y + c_3)\,\mathrm{sech}^2(ax + c_1)\,\mathrm{sech}(by + c_2)}{(4 + \tanh(ax + c_1)\,\mathrm{sech}(by + c_2) + \tanh(t - y + c_3))^2}. \tag{52}$$

A schematic diagram of the instanton solution for the parametric choice $a = 1, b = -1$, $c_1 = c_2 = 0$, $c_3 = 0.5$ is shown in figures 4(*b*)–(*f*) for various time intervals. We can see that the instanton expressed by (52) has a maximum amplitude at $t = 0$ while the amplitude decays exponentially as time $|t| \to \infty$.

4.4. Two-instanton solution

A more general form of (51) is given by

$$\phi_1 = d_1\,\mathrm{sn}(c_1 + a_1x; m_1)\,\mathrm{cn}(b_1y; m_2) + d_2\,\mathrm{sn}(c_2 + a_2x; m_3)\,\mathrm{cn}(b_2y; m_4), \tag{53a}$$

$$F_2 = 4, \qquad F_3 = d_3\,\mathrm{sn}(c_3 + t - y; m_5). \tag{53b}$$

Then

$$|S|^2 = \frac{f_1}{f_2}. \tag{54}$$

Here $f_1 = |(d_1a_1\,\mathrm{cn}(u_1; m_1)\,\mathrm{dn}(u_1; m_1)\,\mathrm{cn}(b_1y; m_2) + d_1a_2\,\mathrm{cn}(u_2; m_3)\,\mathrm{dn}(u_2; m_3)\,\mathrm{cn}(b_2y; m_4))d_3\,\mathrm{cn}(u_3; m_5)\,\mathrm{dn}(u_3; m_5)|$, $f_2 = (4 + d_1\,\mathrm{sn}(u_1; m_1)\,\mathrm{cn}(b_1y; m_2) + d_2\,\mathrm{sn}(u_2; m_3)\,\mathrm{cn}(b_2y; m_4) + d_3\,\mathrm{sn}(u_3; m_5))^2$, $u_1 = c_1 + a_1x$, $u_2 = c_2 + a_2x$ and $u_3 = c_3 + t - y$. The above periodic solution for the parametric choices $a_1 = b_1 = a_2 = 1$, $b_2 = -1$, $d_1 = d_2 = d_3 = 1$, $c_1 = c_3 = -5$, $c_2 = 0$, $m_1 = 0.2$, $m_2 = 0.3$, $m_3 = 0.4$, $m_4 = 0.2$, $m_5 = 0.3$ is shown in figure 5(*a*).

As $m_1, m_2, m_3, m_4, m_5 \to 1$, equation (54) degenerates into two-instanton solution given by

$$|S|^2 = \frac{(\mathrm{sech}^2(a_1x)\,\mathrm{sech}(b_1y) + \mathrm{sech}^2(a_2x)\,\mathrm{sech}(b_2y))\,\mathrm{sech}^2(t - y)}{(4 + \tanh(a_1x)\,\mathrm{sech}(b_1y) + \tanh(a_2x)\,\mathrm{sech}(b_2y) + \tanh(t - y))^2}. \tag{55}$$

The time evolution of the solution (55) is shown in figures 5(*b*)–(*f*). Choosing the arbitrary constants appropriately, we have one of the instantons having a maximum amplitude at $t = -2$ while the other at $t = 2$ and decay exponentially as time $|t| \to \infty$.

To generalize the above solutions, we choose

$$\phi_1 = \sum_{j=1}^{M} d_j\,\mathrm{sn}(c_j + a_jx; m_j)\,\mathrm{cn}(f_j + b_jy; n_j), \tag{56a}$$

$$F_2 = 4, \qquad F_3 = \sum_{k=1}^{N} d_k\,\mathrm{sn}(c_k + t - y; m_k) \tag{56b}$$

where $a_j, b_j, c_j, d_j, f_j, c_k, d_k$ are arbitrary constants, m_j, n_j and m_k take values between 0 and 1. One can construct multi-instanton solution by choosing all the values of m_j, n_j and m_k to be equal to 1.

4.5. Bounded multiple solitary waves

In expression (35), one can also easily identify bounded multiple solitary waves all moving with the same velocity. For instance, using the Jacobian elliptic function form (45) with $d_2 = 0$ in the limit $m_1, m_3, m_4 \to 1$, one can obtain multiple solitary waves which are bounded. Figure 6 displays the structure of a two-soliton solution expressed by

$$|S|^2 = \frac{1}{3}\frac{\mathrm{sech}^2\frac{x+y}{3}[\mathrm{sech}^2(t - y + 5) + 2\,\mathrm{sech}^2(t - y - 5)]}{\left[\tanh\frac{x+y}{3} + 8 + \tanh(t - y + 5) + 2\tanh(t - y - 5)\right]^2}, \tag{57}$$

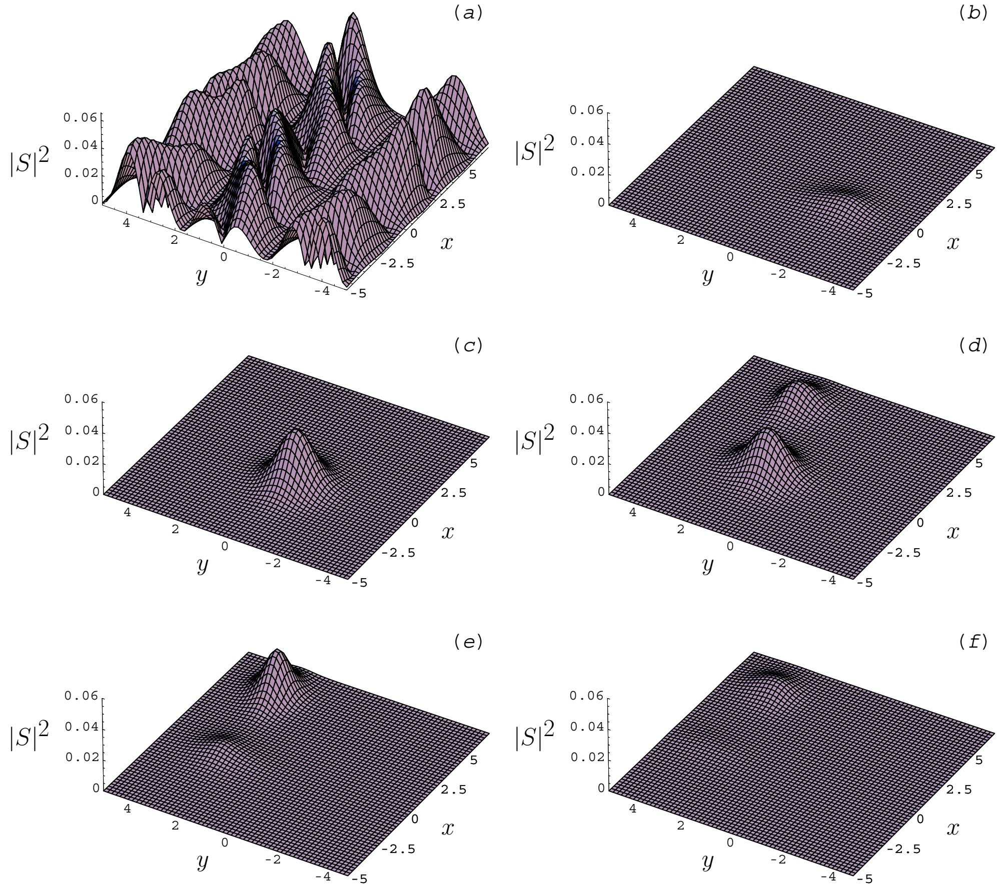

Figure 5. (*a*) Elliptic function solution (54) and two-instanton solution (55) at time intervals (*b*) $t = -3$, (*c*) $t = -1$, (*d*) $t = 1$, (*e*) $t = 3$ and (*f*) $t = 4$.

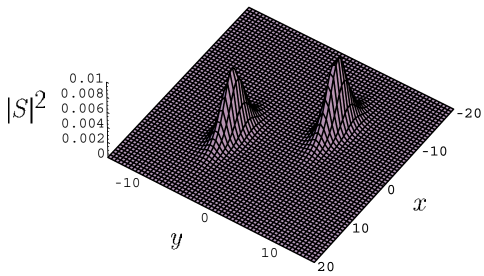

Figure 6. Bounded two-soliton solution.

which corresponds to the selections (see equation (45))

$$\phi_1 = \frac{1}{2} \tanh \frac{x+y}{3}, \qquad F_2 = 4, \qquad F_3 = \frac{1}{2} \tanh(t-y+5) + \tanh(t-y-5). \tag{58}$$

The figure shows that one of the solitary waves follows the other one but both are travelling with equal velocity. Hence, there will not be any interaction between them.

Finally, one can obtain other interesting classes of solutions for different choices of the arbitrary functions in equations (34), (35) and (31).

5. Conclusion

In summary, we have investigated the singularity structure of the (2+1)-dimensional LSRI equation and confirmed that it satisfies the Painlevé property. The Painlevé truncation approach has been used to construct successfully a very wide class of solutions of the (2+1)-dimensional LSRI equation. The rich solution structure of the LSRI equation is revealed because of the entrance of three arbitrary functions in (34) and (31). Especially, Jacobian elliptic function periodic wave solutions and three special localized structures, namely dromion, dromion-type instanton and bounded dromion solutions are given explicitly. However, more general multiple non-bounded dromion solutions whose phase velocities differ from each other have not yet been obtained from the present approach. It appears that one has to solve equation (22) for more general solutions than the form (23) presented in this paper in order to deduce more general solution. This is an open problem at present.

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Appendix A. One dromion solution through Hirota bilinearization

Here we briefly point out how the (1, 1) dromion solution can be obtained through the Hirota bilinearization method [8]. To bilinearize equation (1), we make the transformation

$$S = \frac{g}{f}, \qquad L = 2(\log f)_{xx}, \tag{A.1}$$

which can be identified from the Painlevé analysis in section 2. The resultant bilinear form is given by

$$\left(\mathrm{i}(D_t + D_y) + D_x^2\right)g \cdot f = 0, \tag{A.2a}$$

$$D_x D_t f \cdot f = 2gg^* \tag{A.2b}$$

where D's are the usual Hirota operators. To generate a (1, 1) dromion, one considers the ansatz

$$f = 1 + \exp(\psi_1 + \psi_1^*) + \exp(\psi_2 + \psi_2^*) + M \exp(\psi_1 + \psi_1^* + \psi_2 + \psi_2^*), \tag{A.3}$$

where

$$\psi_1 = px + qy, \tag{A.4a}$$

$$\psi_2 = \lambda y - \Omega t. \tag{A.4b}$$

Here M is a real constant and p, Ω, λ and q are complex constants. Substituting (A.3) into (A.2), we obtain

$$g = \rho \exp(\psi_1 + \psi_2), \tag{A.5a}$$

$$|\rho|^2 = (p + p^*)(q + q^*)(1 - M) \tag{A.5b}$$

and also the conditions $M < 1, q = \mathrm{i}p^2$ and $\Omega = \lambda$. This is a special case of the dromion we have obtained in (40) with the constants $c_1 = c_2 = \frac{1}{2}\log\frac{3}{2}$ and by choosing $\psi_{1R} = -(ax + by + c_1)$ and $\psi_{2R} = -(t - y + c_2)$ and $M = \frac{1}{3}$, where ψ_{1R}, ψ_{2R} are the real parts of ψ_1, ψ_2, respectively. Thus, equation (40) contains the solution of Lai and Chow [8] as a special case. No higher order solution has been constructed by this method.

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